

Gamma Function

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

$$\boxed{1} \quad \Gamma(1) = 1 \qquad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\boxed{2} \quad \Gamma(n) = (n-1)!$$

$$\boxed{3} \quad \Gamma(x) \cdot \Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

$$\boxed{4} \quad \Gamma(\alpha+1) = \alpha \Gamma(\alpha)$$

$$\boxed{Ex} \quad \text{show that } \Gamma(\alpha+1) = \alpha \Gamma(\alpha)$$

Sol

$$\Gamma(\alpha+1) = \int_0^{\infty} x^{\alpha} e^{-x} dx$$

$$u = x^{\alpha}$$

$$dv = e^{-x} dx$$

$$du = \alpha x^{\alpha-1} dx$$

$$\xrightarrow{\quad} v = -e^{-x}$$

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$$\Gamma(\alpha+1) = -x^\alpha e^{-x} \Big|_0^\infty + \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx$$

$$\rightarrow e^{-\infty} = 0, \quad e^0 = 1$$

$$\therefore \Gamma(\alpha+1) = \alpha \Gamma(\alpha)$$

ex:2 show that $\Gamma(n) = (n-1)!$, n integer

Sol

$$\Gamma(n) = (n-1) \Gamma(n-1) = (n-1)(n-2) \Gamma(n-2)$$

$$= (n-1)(n-2)(n-3) \Gamma(n-3)$$

$$= (n-1)(n-2)(n-3) \dots \Gamma(1)$$

$$\Gamma(1) = \int_0^\infty e^{-x} dx = -e^{-x} \Big|_0^\infty = -[e^{-\infty} - e^0] = 1$$

$$\Gamma(n) = (n-1)!$$

~~تمت~~ ~~البرهان~~ ~~بسهولة~~ QED

1 كيفية حساب Γ (رقم)

كسر موجب

عدد طبيعي موجب

عدد سالب

* نغير قيمة " Γ "

* نطرح منه واحد

* نستخدم

حتى تنحصر بين "1,0"

ونضعه داخل المقروب

$$\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}$$

باستخدام

$$\Gamma(n) = (n-1)!$$

$$\Gamma(\alpha+1) = \alpha \Gamma(\alpha)$$

Ex: Evaluate

① $\Gamma\left(\frac{5}{2}\right)$

② $\Gamma\left(-\frac{5}{2}\right)$

③ $\Gamma(7)$

Sol

① $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right) \left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)$

$$= \frac{3}{4} \sqrt{\pi}$$

② $\Gamma\left(-\frac{5}{2}\right) = \frac{\Gamma\left(-\frac{3}{2}\right)}{-\frac{5}{2}}$

نرود (1) ونقسم على القيمة

$$= \frac{-2}{5} \left[\frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}} \right] = \frac{4}{15} \left(\frac{\Gamma\left(\frac{1}{2}\right)}{-\frac{1}{2}} \right)$$

$$= \frac{-8}{15} \pi$$

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Ex show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Sol

$$\Gamma(n) = \int_0^{\infty} u^{n-1} e^{-u} du$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} u^{-\frac{1}{2}} e^{-u} du$$

Put

$$u = x^2$$

$$du = 2x dx$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{\infty} x^{-1} e^{-x^2} \times 2x dx$$

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^{\infty} e^{-x^2} dx$$

$$\rightarrow \Gamma\left(\frac{1}{2}\right)^2 = \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)$$

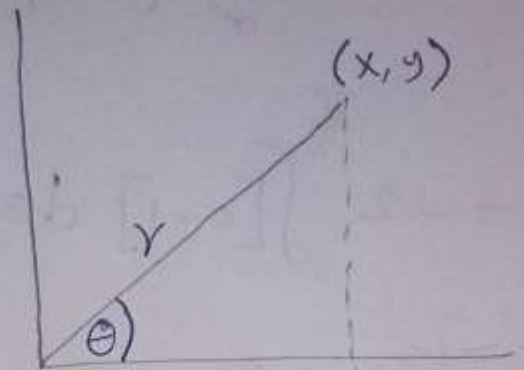
$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \left(2 \int_0^{\infty} e^{-x^2} dx\right) \left(2 \int_0^{\infty} e^{-y^2} dy\right)$$

$$= 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy$$

نستخدم الإحداثيات القطبية

$$x = r \cos \theta$$

$$y = r \sin \theta$$

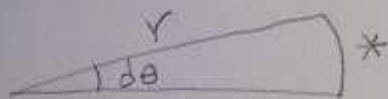
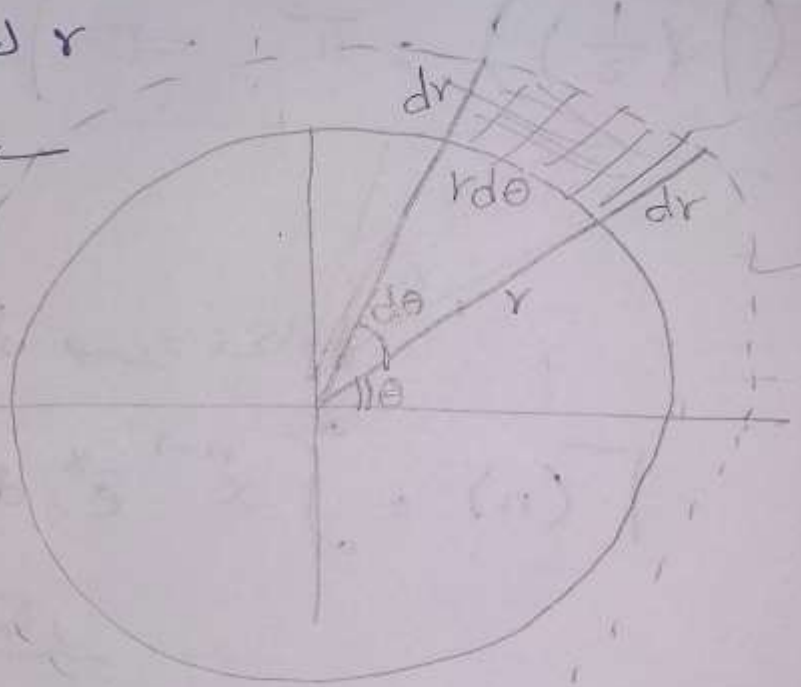


أحدها قسما بلغ المنطقة (الخط)

r لخدما نتج دائرة نصف قطرها r

نزيد الخط بقيمة (dr) وندوره

نتج دائرة أكبر.



$$\sin d\theta = \frac{*}{r}$$

$$* = r \sin d\theta = r d\theta \quad 0 < r < \infty, 0 < \theta < \frac{\pi}{2}$$

$$\left(\Gamma\left(\frac{1}{2}\right) \right)^2 = 4 \int_0^{\frac{\pi}{2}} \left(\int_0^{\infty} e^{-r^2} r dr \right) d\theta$$

~~underline~~

$$= \frac{4}{-2} \int_0^{\frac{\pi}{2}} \left(\int_0^{\infty} e^{-r^2} - 2r dr \right) d\theta$$

$$= -2 \int_0^{\frac{\pi}{2}} \left(e^{-r^2} \Big|_0^{\infty} \right) d\theta$$

$$= -2 \int_0^{\frac{\pi}{2}} [0 - 1] d\theta = 2 \theta \Big|_0^{\frac{\pi}{2}}$$

$$\left(\Gamma\left(\frac{1}{2}\right) \right)^2 = \pi \Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

[2] أفكار التكامل

* نحاول تحويل رأس المسألة إلى صورة تكامل

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

الأفكار

(a) اللوغاريتم e غير جامد \Leftarrow نوع ~~ال~~ ما فوه e بسالب
رمز

$$(ex) \int x^2 e^{-5x} dx$$

$$(ex) \rightarrow a$$

دالة $f(x)$

(b) يظهر داخل التكامل (رقم) $a =$

$$\text{Let: } a = e^{f(x)} = e^{\ln a^{f(x)}} = e^{f(x) \ln a}$$

(c) يظهر داخل التكامل $\ln x$ والحدود من 0 إلى 1.

نفعل $x = e^{-t} \quad \Leftarrow \ln x = -t$

$\ln 0 = -\infty, \ln 1 = 0$

EX Evaluate

- ① $\int_0^{\infty} x^3 e^{-2x} \cosh x \, dx$
- ② $\int_3^{\infty} e^{6x-x^2} \, dx$
- ③ $\int_0^{\infty} \sqrt{x} e^{-x^3} \, dx$
- ④ $\int_0^1 \sqrt{\ln(\frac{1}{x})} \, dx$
- ⑤ $\int_0^{\infty} (1+\sqrt{x})^2 e^{-x} \, dx$
- ⑥ $\int_0^{\infty} x^2 e^{-x/3} \, dx$
- ⑦ $\int_{-\infty}^{\infty} (3x - e^x) e^{-x} \, dx$
- ⑧ $\int_0^{\frac{1}{2}} x^{m-1} \ln(\frac{1}{2x}) \, dx$

solution

① $I = \int_0^{\infty} x^3 e^{-2x} \left[\frac{e^x + e^{-x}}{2} \right] dx$

$$= \underbrace{\frac{1}{2} \int_0^{\infty} x^3 e^{-x} \, dx}_{I_1} + \frac{1}{2} \int_0^{\infty} x^3 e^{-3x} \, dx$$

$\downarrow \quad \quad \quad \downarrow$
 $I_1 \quad \quad \quad I_2$

$$I_1 = \frac{1}{2} \int_0^{\infty} x^3 e^{-x} dx = \frac{1}{2} \Gamma(4) = \frac{1}{2} (3!)$$

$$I_2 = \frac{1}{2} \int_0^{\infty} x^3 e^{-3x} dx$$

$$u = 3x \rightarrow x = \frac{u}{3} \rightarrow dx = \frac{du}{3}$$

$$I = \frac{1}{2} \int_0^{\infty} \frac{u^3}{27} e^{-u} \times \frac{du}{3}$$

$$= \frac{1}{2 \cdot 3^4} \int_0^{\infty} u^3 e^{-u} du = \frac{1}{2 \cdot 3^4} \Gamma(4) = \frac{3!}{2 \times 3^4}$$

$$I = I_1 + I_2 = \frac{1}{2} (3!) + \frac{3!}{2 \times 3^4}$$

④

$$I = \int_0^1 \sqrt{\ln\left(\frac{1}{x}\right)} dx = \int_0^1 \sqrt{\ln x^{-1}} = \int_0^1 \sqrt{-\ln x} dx$$

$$\ln x = -u \quad , \quad x = e^{-u} \Rightarrow dx = -e^{-u} du$$

$$\text{at } x=0 \Rightarrow \ln 0 = -u \Rightarrow u = \infty$$

$$\text{at } x=1 \Rightarrow \ln 1 = -u \Rightarrow u=0$$

$$I = \int_{-\infty}^0 u^{\frac{1}{2}} e^{-u} du = \int_0^{\infty} u^{\frac{1}{2}} e^{-u} du$$

$$= \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$$

$$\boxed{7} \quad I = \int_{-\infty}^{\infty} \frac{(3x - e^x)}{e^x} dx$$

$$= \int_{-\infty}^{\infty} \frac{3x}{e^x} \cdot \frac{-e^x}{e^x} dx = \int_{-\infty}^{\infty} (e^x)^3 e^{-e^x} dx$$

$$u = e^x \quad dx = \frac{du}{u}$$

$$e^{-\infty} = 0, \quad e^{\infty} = \infty$$

$$I = \int_0^{\infty} u^3 e^{-u} \frac{du}{u} = \int_0^{\infty} u^2 e^{-u} du$$

$$= \Gamma(3) = 2!$$

$\boxed{9}$ Lec 21